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APPLYING SCREW THEORY TO ROBOT DYNAMICS*

J.M. SELIG

A conceptually simple approach is developed to describe rigid-body dynamics. A brief exposition of screw calculations is given, with the main emphasis on the geometrical and algebraic properties of these objects. It is shown how to describe the kinematics and dynamics of a completely rigid body using screw variables. The corresponding equations are derived. This technique is generalized to systems of completely rigid bodies joined sequentially by hinges. The Lagrange equations are derived for a six-link manipulator controlled by torques acting along the axes of its hinges.

Earlier papers (in particular /1, 2/) tried to use a dual-vector formulation without explaining its geometrical nature. The geometrical description of classical mechanics /3, 4/ is used below. The advantage of this approach is that the corresponding quantities are well-defined geometrical objects: scalars, vectors, tensors etc. The key point of the paper is the recognition that velocities and momenta are different geometrical objects.

1. ***Screw theory and Lie algebras.*** We begin with a short review of the elements of screw theory and their connections with Lie algebras; the details can be found in /5-8/. The motion of a rigid body is usually described by a (4x4) matrix of the form

$$\begin{Vmatrix} R & x \\ 0 & 1 \end{Vmatrix}$$

where x is the translation vector and R is a (3x3) rotation matrix, i.e. an othogonal matrix of special form. The set of all rigid body motions forms the Euclidean group $E(3)$ in three-dimensional space. These (4x4) matrices generate a group representation which enables one to compute the change of the state vector under "rigid" body motions (rotations and translations).

We will consider some motion of the rigid body. At every instant of time the configuration is specified by a group element, and finite motion is specified by a curve on the group manifold.

We will transfer to infinitesimal group elements. They can be treated as tangent vectors to the group at some point. Computations can be carried out using any matrix representation of the group. We will assume that

$$\gamma : t \mapsto \begin{Vmatrix} R(t) & x(t) \\ 0 & 1 \end{Vmatrix}$$

is a curve on $E(3)$ parameterized by t. The derivative at any point $g = \gamma(t_0)$ is simply equal to $d\gamma(t_0)/dt$. If g is the identity element, then one can say more about the tangent vectors at that point. In particular, for $E(3)$ we have $R^TR = 1$ because of the orthogonality of the submatrix R. Differentiating this expression, we have $R^{\cdot T} R + R^TR^{\cdot} = 0$. For the identity element the formula simplifies: $R^{\cdot T} + R^{\cdot} = 0$, so that the matrix $R^{\cdot}$ is antisymmetric. Thus the tangent vector to the identity element has the form

$$s = \begin{Vmatrix} \Omega & v \\ 0 & 0 \end{Vmatrix}$$

where Ω is an antisymmetric 3x3 matrix and v is a three-dimensional vector.

These matrices form a vector space. The product of two antisymmetric matrices does not in general give a matrix of the same form, but the commutator is equal to the tangent vector to the identity element. The commutator of two vectors is defined by $[s_1, s_2] = s_1s_2 - s_2s_1$.

The tangent space to the group at the identity element thereby possesses the structure of a Lie algebra, which we shall denote by $e(3)$. The commutator is similar to multiplication,

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but is not associative. It is instead governed by the Jacobi identity:

$$[s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0$$

From Shale's theorem it is clear that the finite screws described by Ball /9/ are the rigid body motions considered above. Moveover, Ball's instantaneous screws are nothing other than the elements of the Lie algebra. The relationship becomes clear after representing the elements of the Lie algebra as six-dimensional vectors:

$$s = \begin{Vmatrix} \Omega & v \\ 0 & 0 \end{Vmatrix} \equiv \begin{Vmatrix} \omega \\ v \end{Vmatrix}$$

Because Ω is an antisymmetric matrix one can use the standard antisymmetric tensor ε_{ijk} so that in some Cartesian coordinate system $\Omega_{ij} = -\varepsilon_{ijk}\omega_k$, or more explicitly,

$$\Omega = \begin{Vmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{Vmatrix}$$

It is also convenient to represent screws by dual vectors

$$s = \omega + \varepsilon v$$

Here ε is a formal symbol, called the dual unit, which commutes with vectors and satisfies the condition $\varepsilon^2 = 0$.

Easy computations show that

$$[s_1, s_2] = s_1 \wedge s_2 = \omega_1 \wedge \omega_2 + \varepsilon(\omega_1 \wedge v_2 + v_1 \wedge \omega_2) = (\omega_1 + \varepsilon v_1) \wedge (\omega_2 + \varepsilon v_2)$$

This means that the commutator of the Lie algebra corresponds to the dual vector product $\wedge$. (It is unnecessary to use different notation for the dual and standard vector products because the sense should always be clear from the context).

The velocity screw of a rigid body can be computed as follows. Suppose the motion of a rigid body is given, as before, by the relation $\gamma(t)$ where t is now the time. The derivative with respect to time is $\gamma^{\cdot}(t)$. At any instant $t = t_0$ this is a tangent vector to the group, but not necessarily a screw. In order to turn it into a screw, it is necessary to move it to the unit element by multiplying it on the right by the inverse element $\gamma(t)^{-1}$. Thus the velocity screw of the body at time t_0 is equal to

$$s(t_0) = \gamma^{\cdot}(t_0)\gamma^{-1}(t_0) = \begin{Vmatrix} R^{\cdot}(t_0)R^T(t_0) & -R^{\cdot}(t_0)R^T(t_0)x(t_0) + x^{\cdot}(t_0) \\ 0 & 0 \end{Vmatrix}$$

In terms of dual vectors we find that

$$s(t_0) = \omega(t_0) + \varepsilon(v(t_0) - \omega(t_0) \wedge x(t_0))$$

where ω is the usual angular velocity of the rigid body, similar to the case when one considers only the rotation relative to some fixed point (see e.g., /10, p.8/.

Every Lie group G has a natural adjoint representation that acts on its Lie algebra and appears as a result of considering conjugates in the group. The maps $f_g: G \to G$, where $f_g(h) = ghg^{-1}$, are smooth for all $g, h \in G$. They all turn the identity element into itself, so that the Jacobian of each map at the unit element is a linar map on the Lie algebra. Because $f_g f_{g'} = f_{gg'}$, the Jacobian specifies a linear representation of the group G.

Using the original $E(3)$ representation one can compute the adjoint representation:

$$\begin{Vmatrix} \Omega' & v' \\ 0 & 0 \end{Vmatrix} = \begin{Vmatrix} R & x \\ 0 & 1 \end{Vmatrix} \begin{Vmatrix} \Omega & v \\ 0 & 0 \end{Vmatrix} \begin{Vmatrix} R^T & -R^T x \\ 0 & 1 \end{Vmatrix} = \begin{Vmatrix} R\Omega R^T & -R\Omega R^T x + Rv \\ 0 & 0 \end{Vmatrix}$$

In terms of six-dimensional vectors this equality can be written as

$$\begin{Vmatrix} \omega' \\ v' \end{Vmatrix} = \begin{Vmatrix} R & 0 \\ XR & R \end{Vmatrix} \begin{Vmatrix} \omega \\ v \end{Vmatrix}$$

where $X_{ij} = -\varepsilon_{ijk}x_k$ while R is the same quantity as before. This gives a description of screw transformations under "rigid" coordinate changes.

One can map the elements of the Lie algebra into the original group using the exponential map

$$s \mapsto e^s = 1 + s + s^2/2 + \ldots + s^n/n! + \ldots$$

If one uses a matrix representation of the Lie algebra, the degree of s can be interpreted as the degree of the matrix. The result of the exponential map is a matrix in the corresponding group represenation. However, irrespective of whatever representation is used, one obtains the same element of the group. Thus the map has meaning even if it is not associated with a definite representation.

We choose some element $s \in e(3)$. Then the set of group elements of the form $e^{\theta s}$ for all scalars θ forms a subgroup of the group $E(3)$. Physically this subgroup is the symmetry group for a lower Reuleaux pair with one degree of freedom /9/. It follows from this that rigid motions relative to the hinge under consideration reduce to a one-parameter subgroup.

We will further assume that the hinge parameter increases at a constant rate. The relative motion of the two sides of a hinge is given by the quantity $g(t) = e^{ts}$ and the velocity screw of such motion has the form $g^{\cdot}g^{-1} = se^{ts}e^{-ts} = s$, i.e. is constant.

2. *Momentum and inertia.* A standard characteristic of rigid body motion is a screw $a = p + \varepsilon M$, which from now on is called the momentum. The kinetic energy of the body can be written as

$$E_k = {}^1/_2\, s \circ a = {}^1/_2\,(p\,v + M\omega)$$

This equation appears to be due to Ball /9/ and, although formally correct, gives a false representation of the nature of the momentum. The adequacy of this equation is explained by a chance property of three-dimensional space.

In modern mechanics, momentum is taken to be a linear function of the velocities: $r^*: e(3) \to \mathbf{R}$. The space of all such functions forms a vector space of the same dimensionality as the original velocity vector space. This function vector space is usually called the dual vector space. In order to avoid confusion with dual numbers, the six-dimensional velocity vectors will be called screws, and the vector momenta coscrews. In more traditional language screws are called covariant vectors, and coscrews are contravariant vectors. The vector space of coscrews will be denoted by $e^*(3)$.

By analogy with screws one can represent a coscrew as a column-vector $r^* = (M, p)^T$, where M and p are the angular and linear momenta respectively. The expression $r^*(s)$ can be written as a matrix product

$$r^*(s) = (M^T, p^T)\begin{Vmatrix} \omega \\ v \end{Vmatrix} = 2E_k$$

It is now clear that screws and coscrews are different objects, because they change differently under "rigid" coordinate transformations. In new coordinates $s' = Hs$, where H is a (6x6) block matrix of the form

$$H = \begin{Vmatrix} R & 0 \\ XR & R \end{Vmatrix}$$

Here R is a rotation matrix and $X_{ij} = -\varepsilon_{ijk}x_k$. This is in fact the adjoint representation of the group in its Lie algebra.

In order that the scalar quantity $r^*(s)$ remain constant, it is necessary for $r^{*\prime} = (H^T)^{-1} r^*$. Unlike the usual pure-rotation cases, the matrix H is not orthogonal, so that $(H^T)^{-1} \neq H$. In fact

$$(H^T)^{-1} = \begin{Vmatrix} R & XR \\ 0 & R \end{Vmatrix}$$

In this form the concept of a coscrew is not all that new. For example, for a robot with a six hinges the rows of the inverse Jacobi matrix are coscrews /11/. They are also encountered in hybrid robot control problems /12/, because "efforts" are also coscrews. It is important to note that screws and coscrews are indeed different, because even though the two vector spaces are isomorphic, there is no natural isomorphism.

Many authors have tried to represent rigid body dynamics using dual vector formalism. This is an attractive idea because the algebra of dual vectors is more compact. An awkwardness is caused by the property that if one considers velocities and momenta as dual vectors, it is difficult to interpret momenta of inertia, and usually this leads to awkward notation and expressions.

We will now look at the algebraic properties of coscrews. Suppose an orthonormal vector basis is chosen in $\mathbf{R}^3$. The corresponding choice of basis vectors in $e(3)$ is denoted by $i, j, k, \varepsilon i, \varepsilon j$ and εk. Then in dual notation the screw can be written as

$$s = \omega_x i + \omega_y j + \omega_z k + v_x \varepsilon i + v_y \varepsilon^j + v_z \varepsilon k$$

The corresponding coscrew basis is $i^*, j^*, k^*, \varepsilon_i^*, \varepsilon_j^*$ and ε_k^*, where

$$l^*(n) = \varepsilon_l^*(\varepsilon n) = \begin{cases} 1, & l = n \\ 0, & l \neq n \end{cases}, l, n \in \{i, j, k\}$$

and all non-diagonal terms are equal to zero.

It follows from this that the coscrew can be written as

$$r^* = M_x i^* + M_y j^* + M_z k^* + p_x \varepsilon_i^* + p_y \varepsilon j^* + p_z \varepsilon_k^*$$

We remark that $e^*(3)$ is a vector space over the reals, but not over dual numbers. The possibility of vector multiplication of momenta is lost, but in return we obtain a new bilinear operation $\{\ \}$: $e^*(3) \times e(3) \to e^*(3)$. It is obtained from the screw Lie bracket using the relation

$$\{r^*, s_1\}(s_2) = r^*(s_1 \wedge s_2)$$

For example, $\{\varepsilon_i^*, \varepsilon j\} = -k^*$ because

$$\{\varepsilon_i^*, \varepsilon j\}(k) = \varepsilon_i^*(\varepsilon j \wedge k) = \varepsilon_i^*(-\varepsilon i) = -1$$

However,

$$\{\varepsilon_i^*, \varepsilon j\}(l) = \varepsilon_i^*(\varepsilon j \wedge l) = 0$$

for all basis elements $l \neq k$.

In column-vector terminology one can show that

$$\{r^*, s\} = \left\{\begin{Vmatrix} M \\ p \end{Vmatrix}, \quad \begin{Vmatrix} \omega \\ v \end{Vmatrix}\right\} = \begin{Vmatrix} \omega \wedge M + v \wedge p \\ \omega \wedge p \end{Vmatrix}$$

The latter expression is a vector product of two screws whose first and last three components are represented. However, the above relates to any coordinate system.

The question arises of the interpretation of moments of inertia after the separation of velocities and momenta. The answer lies in interpreting moments inertia as operators (tensor) turning velocities into momenta. The linear isomorphism $K: e(3) \to e^*(3)$ corresponds to the inertia operator. The operator K can be represented as a symmetric tensor or as a symmetric (6x6) matrix. Thus the correct method of obtaining coscrews from screws is given by the equality $r^* = Ks$. We still have not chosen K uniquely. However, for a definite rigid body K is precisely specified by equations from elementary mechanics.

$$M = A\omega + m(c \wedge V), \ p = mv + m(\omega \wedge c)$$

where m is the mass of the body, c is the position of the centre of mass and A is a (3x3) inertia tensor. We then have the formula

$$K = \begin{Vmatrix} A & mC \\ mC^T & mI \end{Vmatrix}$$

As usual, I is the unit (3x3) tensor and $C_{ij} = -\varepsilon_{ijk} c_k$. The kinetic energy of the body is given by the relation $E_k = (1/2)\, Ks(s) = (1/2)\, s^T K s$, where s is interpreted as a column-vector. From this one can discern that the coordinate change given by the (6x6) matrix specified above leads to the following change in the inertia tensor:

$$K' = (H^T)^{-1} K H^{-1}$$

This equation reflects a combination of the tensorial properties of the (3x3) inertia tensor and Steiner's parallel axes theorem. It can be thought of as an oblique axes theorem.

In order to see the connection with Steiner's theorem, we consider the case when H is a pure translation and the origin of coordinates is at the body's centre of mass. In this case the computation simplifies because

$$K = \begin{Vmatrix} A & 0 \\ 0 & mI \end{Vmatrix}, \qquad H = \begin{Vmatrix} I & 0 \\ X & I \end{Vmatrix}$$

and so

$$K' = \left\| \begin{matrix} I & X \\ 0 & I \end{matrix} \right\| \left\| \begin{matrix} A & 0 \\ 0 & mI \end{matrix} \right\| \left\| \begin{matrix} I & 0 \\ -X & I \end{matrix} \right\| = \left\| \begin{matrix} A - mX^2 & mX \\ -mX & mI \end{matrix} \right\|$$

Computation shows that $X^2 = xx^T - x^T xI$ where xx^T is the exterior or dyadic product. For comparison we refer the reader to the formulation of Steiner's theorem in the textbooks, (for example /10 p.85/).

3. ***The Newton-Euler equations.*** Having now introduced a successful notation, one can obtain the equations of motion for an isolated rigid body, imitating the standard derivation of the equations of motion with a fixed point. We begin by comparing time derivatives in a static inertial coordinate system Σ and in a system Σ' attached to the rigid body. At each instant of time there exists a rigid transformation $H(t)$ taking screw components from system Σ to Σ'.

Consider an arbitrary screw q or coscrew m^*. Time derivatives in the chosen coordinate systems are written as Dq and $D'q$. Their relations are given by the Coriolis theorem (/10 p.10/).

In coordinate system Σ the screw q can be written as

$$q = \sigma_x i + \sigma_y j + \sigma_z k + u_x \varepsilon i + u_y \varepsilon j + u_z \varepsilon k$$

In the inertial coordinate system the screw time derivative Dq is obtained by componentwise differentiation. In Σ' coordinates we have

$$q = \sigma_x' i + \sigma_y' j + \sigma_z' k + u_x' \varepsilon i + u_y' \varepsilon j + u_z' \varepsilon k$$

These new screw components are given by the relation

$$\left\| \begin{matrix} \sigma' \\ u' \end{matrix} \right\| = H \left\| \begin{matrix} \sigma \\ u \end{matrix} \right\|$$

i.e. the relative motion of the coordinate systems is "rigid". Differentiating the latter equality with respect to time, we obtain

$$\left\| \begin{matrix} \sigma'' \\ u'' \end{matrix} \right\| = H \left\| \begin{matrix} \sigma' \\ u' \end{matrix} \right\| + H' \left\| \begin{matrix} \sigma \\ u \end{matrix} \right\|$$

In terms of time derivatives in the coordinate system this result can be written as

$$D'q = Dq + H'H^{-1}q$$

Using the expression for H from Sect.2 we have

$$H'H^{-1} = \left\| \begin{matrix} R'R^T & 0 \\ X' + XR'R^T - R'R^T X & R'R^T \end{matrix} \right\|$$

$$H'H^{-1}q = s \wedge q, \quad s = \left\| \begin{matrix} \omega \\ x' + x \wedge \omega \end{matrix} \right\|$$

(where s is the velocity screw of coordinate system Σ' relative to Σ, ω is the relative angular velocity of the two coordinate systems, and x is the position vector of the origin of Σ' in Σ). Thus the relation between the time derivatives has the form

$$D'q = Dq + s \wedge q$$

Similarly, for coscrews we have

$$D'm^* = Dr^* + \{r^*, s\}$$

We can now derive the equations of motion. To do this it is sufficient to repeat the derivation of Euler's equations (/4, p.143/). Suppose $m^* = Ks$ is the momentum coscrew of the rigid body. Then from Newton's laws in the inertial coordinate system we have $Dm^* = f^*$, where f^* is a general "effort" applied to the body: $f^* = (T, F)^T$, (where T is the resultant torque and F is the resultant force acting on the body). From the Coriolis theorem

$$D'm^* = \{m^*, s\} + f^*$$

The equation of motion of a rigid body has thus been obtained in Newton-Euler form. We will turn it into a differential equation for s, because $m^* = Ks$, while D' is the derivative in the comoving coordinate coordinate system so that $D'K = 0$. As a result

$$Ks' = \{Ks, s\} + f^*$$

From this it is clear how to combine the equations for several rigid bodies and improve on the usual recursive formulation of robot dynamics which eliminates the reactions at the hinges. However, some caution is necessary here because the screw velocities are represented in coordinates tied to the body (or a link for a robot).

4. Derivatives for kinematic chains. It was shown above that the time derivatives for screws or coscrews in different coordinate systems are connected by simple geometrical relations. This is also true for other derivatives if the kinematics are known. For an open kinematic chain, which is encountered in a six-link robot, the situation is particularly simple.

We assume that $s_1(t), s_2(t), \ldots, s_6(t)$ are the screws of the robot hinges. The change of the position and orientation of the i-th link relative to its original position at time $t = 0$ is determined by some rigid map H_i. The direct kinematic map is represented as a product of exponential functions of angular variables

$$H_i = \exp(\theta_1 s_1(0)) \exp(\theta_2 s_2(0)) \ldots \exp(\theta_i s_i(0))$$

The corresponding velocity screw is determined by the time derivative of this expression (see /1/), but can also be written in terms of hinge velocities

$$q_i^{\cdot} = H_i^{\cdot} H_i^{-1} = J_i \theta^{\cdot}$$

Here $\theta^{\cdot}$ is the six-dimensional vector of the hinge velocities $\theta_1^{\cdot}, \ldots, \theta_6^{\cdot}$, and the matrix J_i is the Jacobian of the direct kinematic map. Its value is easy to estimate, because $H_i = I$ at $t = 0$. Hence

$$H_i^{\cdot}(0) H_i^{-1}(0) = \theta_1^{\cdot}(0) s_1(0) + \theta_2^{\cdot}(0) s_2(0) + \ldots + \theta_i^{\cdot}(0) s_i(0)$$

Here we have implicitly used the matrix representation for screws, but this is unimportant in view of the remarks made at the end of Sect.1. The screws can be represented as six-dimensional vectors. Then the columns of the Jacobi matrix will have the form $J_i = (s_1(0), s_2(0), \ldots, s_i(0), 0, \ldots, 0)$. The Jacobians are (6x6) matrices and J is the usual manipulator Jacobian. No special assumptions are made about the initial configuration. Then, in general, the Jacobian can be written as

$$J_i(t) = (s_1(t), s_2(t), \ldots, s_i(t), 0, \ldots, 0)$$

The columns of the Jacobian are the current values of the hinge screws. In order to simplify the notation in the following, the explicit time dependence will be omitted.

The partial derivatives can also be computed. For example,

$$\frac{\partial}{\partial\theta_j} s_i = \begin{cases} s_j \wedge s_i, & j \leqslant i \\ 0, & j > i \end{cases}, \quad \frac{\partial}{\partial\theta_j} q_i^{\cdot} = \begin{cases} s_j \wedge (q_i^{\cdot} - q_j^{\cdot}), & j \leqslant i \\ 0, & j \geqslant i \end{cases}$$

because the group adjoint representation gives

$$s_i(t) = \exp(\theta_1 s_1(0)) \ldots \exp(\theta_{i-1}(0) s_{i-1}(0)) s_i(0) \exp(-\theta_{i-1} s_{i-1}(0)) \times \cdots \\ \times \exp(-\theta_1 s_1(0))$$

For the coscrews one can use the coadjoint group representation, so that for the coscrew momentum derivatives for the i-th link we obtain

$$\frac{\partial}{\partial\theta_j} m_i^* = \begin{cases} \{m_i^*, \ s_j\}, & j \leqslant i \\ 0, & j > i \end{cases}$$

Because the hinge screws do not depend on $\theta_i^{\cdot}$, their time derivatives, and hence also the Jacobian time derivatives, are easily computed. Bearing in mind that $q_j^{\cdot} = J_j \theta^{\cdot} = \theta_1^{\cdot} s_1 + \ldots + \theta_2^{\cdot} s_2 + \ldots + \theta_j^{\cdot} s_j$, we have

$$\frac{d}{dt} s_i = \sum_{j=1}^{6} \theta_j^{\cdot} \frac{\partial}{\partial\theta_j} s_i = q_i^{\cdot} \wedge s_i$$

Another approach to computing the Jacobian derivatives was given in /13/.

The Jacobian time derivatives are now easily computed. For example, the column of derivatives $J_6^{\cdot}$ has the form

$$J_6^{\cdot} = (q_1^{\cdot} \wedge s_1, q_2^{\cdot} \wedge s_2, \ldots, q_6^{\cdot} \wedge s_6)$$

We consider the time derivatives of the inertia tensor. In Sect.2 we studied the influence of rigid body motion on the inertia tensor. The inertia tensor of the i-th link is given at any time by

$$K_i(t) = (H_i^T)^{-1}(t)K(0)H_i^{-1}(t)$$

Here H_i is the usual product of exponentials, so that at $t = 0$ the time derivative of K_i is equal to

$$K_i^{\cdot}(0) = (-\theta_1^{\cdot} s_1^T - \theta_2^{\cdot} s_2^T - \ldots - \theta_i^{\cdot} s_i^T) K_i(0) + K_i(0)(-s_1\theta_1^{\cdot} - s_2\theta_2^{\cdot} - \ldots - s_i\theta_i)$$

In this equation the screws are to be understood as (6x6) matrices. We write the (6x6) velocity screw of the i-th link as

$$Q_i^{\cdot} = s_1\theta_1^{\cdot} + s_2\theta_2^{\cdot} + \ldots + s_i\theta_i^{\cdot}$$

These computations do not depend on the original position either, and so we have

$$K_i^{\cdot} = -Q_i^{\cdot T} K_i - K_i Q_i^{\cdot}$$

The relation between $q^{\cdot}$ and $Q^{\cdot}$ is given in coordinates as follows:

$$\text{if} \quad q = \begin{Vmatrix} \omega \\ v \end{Vmatrix}, \quad \text{then} \quad Q^{\cdot} = \begin{Vmatrix} \Omega & 0 \\ V & \Omega \end{Vmatrix}, \quad \text{where} \quad V_{ij} = -\varepsilon_{ijk} v_k$$

5. ***Lagrangian mechanics for a six-link robot.*** We consider the kinetic energy E_k of a chain of rigid bodies joined by hinges with one degree of freedom. In view of the remarks made in Sect.3, the generalized kinetic energy of the manipulator can be written as

$$E_k = (1/2) \sum_{i=1}^{6} m_i^*(q_i^{\cdot}) = (1/2) \sum_{i=1}^{6} q_i^{\cdot T} K_i q_i^{\cdot} = (1/2)\, \theta^{\cdot T} \{ \sum_{i=1}^{6} J_i^T K_i J_i \} \theta^{\cdot}$$

where m_i^* is the coscrew momentum of the i-th link.

The conjugate momentum for the coordinate θ_i is computed using the fact that it is already known in terms of the Jacobian's rows. The result is rather simple because most of the terms do not depend on θ_i:

$$\frac{\partial E_k}{\partial \theta_i^{\cdot}} = \frac{1}{2} \frac{\partial}{\partial \theta_i^{\cdot}} \left(\theta^{\cdot T} \left\{ \sum_{i=1}^{b} J_i^T K_i J_i \right\} \theta^{\cdot} \right) = \sum_{j=i}^{6} q_j^{\cdot T} K_j s_i = \sum_{j=i}^{6} m_j^*(s_i)$$

This relation means that the conjugate momentum corresponding to the angle of the i-th hinge can be found by computing the sum of the hinge momenta starting with the i-th, at the i-th hinge screw.

We consider the derivatives of the kinetic energy with respect to the screw angles. Applying the relations found for derivatives of screws and coscrews, we have

$$\frac{\partial E_k}{\partial \theta_i} = \frac{1}{2} \frac{\partial}{\partial \theta_i} \sum_{j=1}^{6} m_j^*(q_j^{\cdot}) = \frac{1}{2} \sum_{j=1}^{6} m_j^*(s_i \wedge q_j) + m_j^*(s_i \wedge (q_j^{\cdot} - q_i^{\cdot})) = \sum_{j=i}^{6} m_j^* \left(s_i \wedge \left(q_j^{\cdot} - \frac{1}{2} q_i^{\cdot} \right) \right)$$

Combining these results, we obtain an expression for the generalized momenta associated with the generalized coordinates θ_i:

$$\frac{d}{dt} \left(\frac{\partial E_k}{\partial \theta_i^{\cdot}} \right) - \frac{\partial E_k}{\partial \theta_i} = \sum_{j=i}^{6} m_j^{*\cdot}(s_i) + m_j^*((q_j^{\cdot} + q_i^{\cdot}) \wedge s_i)$$

In the case when gravity is not taken into account, the latter expression gives the hinge momenta directly. However, as a rule, one must take into account the gravitational contribution to the potential energy. The potential energy can be found if the heights of the centres of mass of the links are known. For convenience we write $\rho_i = (m_i c_i, c_i)$, where m_i is the mass of the i-th link and c_i is the coordinate vector of the centre of mass. If we denote by $\rho_i(0)$ the value of ρ_i in the original configuration, then the later values will be

$$\rho_i = \exp(\theta_1 s_1(0)) \ldots \exp(\theta_i s_i(0)) \rho_i(0)$$

Here the screws are understood to be (4x4) matrices.

We assume that gravity acts in the $-k$ direction. Then the potential energy of the robot can be written as

$$E_p = g e_k^T \sum_{i=1}^{6} \rho_i$$

where $e_k = (k, 0)^T$ is a four-dimensional vector and g is the acceleration due to gravity. The partial derivatives give the values of the torques associated with the hinges:

$$\frac{\partial E_p}{\partial \theta_i} = g e_k{}^T s_i \sum_{j=i}^{6} \rho_j$$

The screw is again taken to be a (4x4) matrix. However one notes that a typical term on the right-hand side can be written with the help of three-dimensional vectors:

$$g e_k{}^T s_i \rho_j = k\,(m_j g \omega_j \wedge c_j + m_j g v_j)$$

The expression simplifies if we use the duality between the screws and the "efforts". One can write the typical term as

$$- W_j{}^*(s_i) = m_j g k\,(\omega_j \wedge c_j + v_i)$$

$$W_j{}^* = \begin{Vmatrix} -m_j g c_j \wedge k \\ -m_j g k \end{Vmatrix}$$

where $W_j{}^*$ is the effort due to gravity acting on the j-th link, with the hinge screws again represented by six-dimensional vectors.

Combining all these results and the Lagrange equations we obtain a relation for the generalized momenta of the hinges:

$$\tau_i = \sum_{j=1}^{6} m_j^{*\cdot}(s_i) + m_j{}^*((q_j{}^{\cdot} + q_i{}^{\cdot}) \wedge s_i) + W_j{}^*(s_i)$$

They can also be written solely in terms of velocity screws and momenta of inertia. To do this one must find the time derivatives of the coscrew momenta. Because $m_j{}^* = K_j q_j{}^{\cdot}$ we have

$$m_j{}^{*\cdot} = K_j{}^{\cdot} q_j{}^{\cdot} + K_j q_j{}^{\cdot\cdot}$$

Using the relations for the time derivatives of the inertia tensor obtained in Sect.4 we have

$$m_j{}^{*\cdot} = K_j q_j{}^{\cdot\cdot} - Q_j{}^{\cdot T} K_j q_j{}^{\cdot}$$

Here we used the equalities $Q^{\cdot} q^{\cdot} = q^{\cdot} \wedge q^{\cdot} = 0$.

The value of the coscrew momentum at the hinge screw is equal to

$$m_j{}^{*\cdot}(s_i) = q_j{}^{\cdot\cdot T} K_j s_i - q_j{}^{\cdot T} K_j (q_j{}^{\cdot} \wedge s_i)$$

Thus in terms of the link velocities the Euler-Lagrange equations take the form

$$\tau_i = \sum_{j=i}^{6} q_j^{\cdot\cdot T} K_j s_i + q_j^{\cdot T} K_j (q_i{}^{\cdot} \wedge s_i) + W_j{}^*(s_i)$$

These equations look particularly simple and elegant. However, they can be written in terms of hinge coordinates, for which we note that $q_i{}^{\cdot\cdot} = J_i \theta^{\cdot\cdot} + J_i \theta^{\cdot}$. After some calculations the equations of motion take the form

$$\tau_i = \sum_{j=i}^{6} \{ \sum_{k=1}^{j} s_i{}^T K_j s_k \theta_k{}^{\cdot\cdot} + \sum_{k=2}^{j} \sum_{e=1}^{k-1} s_i{}^T K_j (s_e \wedge s_k)\, \theta_k{}^{\cdot} \theta_e{}^{\cdot} +$$

$$\sum_{k=1}^{j} \sum_{e=1}^{i-1} s_k{}^T K_j (s_e \wedge s_i) \theta_k{}^{\cdot} \theta_e{}^{\cdot} + W_j{}^* (s_i)\}$$

6. Concluding remarks. The basic aim of this paper has been to show the simple expressions can be obtained for quantities used in robot dynamics. If the robot is a standard chain of rigid bodies, united by hinges with one degree of freedom, these expressions can be computed directly.

When the work was begun, there was hope that the use of screw theory could, from the computational point of view, be preferable to the usual approach. Unfortunately, these hopes were not fulfilled, although comparison is somewhat difficult because it is not totally clear what should be included in the operation count. One could somewhat improve the algorithm being used by introducing a "gravitational screw" $g = (0, gk)^T$, so that $W_j{}^* = K_j g$. Then the potential term in the equations of motion can be included in the term containing the accelerations:

$$\tau_i = \sum_{j=i}^{6} (q_j{}^{\cdot\cdot} - j)^T K_j s_i + q_j{}^{\cdot T} K_j (q_i{}^{\cdot} \wedge s_i)$$

The results obtained are important from the theoretical point of view, because everything written above has a geometrical interpretation. No assumptions about the hinges, other than that each one has a single degree of freedom, have been made. The author also tried to use the smallest number of coordinate systems, usually one. There is hope that this will enable problems connected with robot design to be investigated. For example, how does one design a robot so as to avoid or minimize the dynamical interaction between the links? We assume it is necessary to guarantee that the momentum of the i-th hinge τ_i does not depend on the accelerations of the other hinges. It is easy to verify that this means that $\partial^2 E_k / \partial\theta_i{}^{\cdot} \partial\theta_j{}^{\cdot} = 0$, $i \neq j$. Making use of previously obtained results we have

$$\frac{\partial^2 E_k}{\partial\theta_i{}^{\cdot} \partial\theta_j{}^{\cdot}} = \sum_{k=i}^{j} s_j^T K_k s_i = 0, \quad i \neq j$$

This sets bounds on the possible positions of the hinges and masses.

It is now clear what one must do to perform a systematic study of the problem.

The author also hopes that the results obtained can be extended to the case of elastic hinges and thereby lead to a simplification of the treatment models that are important in practice.

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